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Engwerda, J.C.

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On the open-loop Nash equilibrium in LQ-games

by

Jacob C. Engwerda

Tilburg University
Department of Econometrics
P.O. Box 90153
5000 LE Tilburg
The Netherlands

Abstract

In this paper we consider open-loop Nash equilibria of the linear-quadratic differential game. As well the finite-planning-horizon, the infinite-planning horizon as convergence properties of the finite-planning-horizon equilibrium if the planning horizon is extended to infinity are studied. Particular attention is paid to computational aspects and the scalar case.

Keywords: Linear quadratic games, open-loop Nash equilibrium, solvability conditions, Riccati equations

I. Introduction

The last decade there has been an increasing interest to study several problems in economics using a dynamic game theoretical setting. In particular in the area of environmental economics and macro-economic policy coordination this is a very natural framework to model problems (see e.g. de Zeeuw et al. (1991), Mäler (1992), Kaitala et al. (1992) and Dockner et al. (1985), Tabellini (1986), Fershtman et al. (1987), Petit (1989), Levine et al. (1994), van Aarle et al. (1995), Douven et al (1996)). In, e.g., policy coordination problems usually two basic questions arise i.e., first, are policies coordinated and, second, which information do the participating parties have. Usually both these points are rather unclear and, therefore, strategies for different possible scenarios are calculated and compared with each other. One of these scenarios is the so-called open-loop strategy. This scenario can be interpreted as that the parties simultaneously determine their strategy, next submit their strategies to some authority who then enforces these plans as binding commitments. So, this strategy is based on the assumption that the parties act non-cooperatively and that the only information they have on the model is its present state and the model structure. Obviously, since according this scenario the participating parties can not react to each other's policies, its economic relevance is mostly rather limited. However, as a benchmark to see how much parties can gain by playing other strategies, it plays a fundamental role. Due to its analytic tractability the open-loop Nash equilibrium strategy is in particular very popular for problems where the underlying model can be described by a (set of) linear differential equation(s) and the individual objectives, the parties are striving for, can be approximated by functions which quadratically penalize deviations from some (equilibrium) targets. Under the assumption that the parties only have a finite-planning horizon, this problem was first modeled and solved in a mathematically rigorous way by Starr and Ho in (1969). However, due to some inaccurate formulations it is, even in nowadays literature, an often encountered misunderstanding that this problem always has a unique Nash equilibrium strategy which can be obtained in terms of the solutions of a set of coupled matrix differential equations resembling (but more complicated than) the matrix Riccati equations which arise in optimal control theory. Eisele, who extended the Hilbert space approach of this problem taken by Lukes et al (1971), in (1982) already noted that there are some misleading formulations in the literature. But, probably due to the rather abstract approach he took, this point was not noted in the mainstream literature. So, in other words, there exist situations where the set of coupled matrix differential equations has no solution, whereas the problem does have an equilibrium. We will present such an example here and use the more direct simple Hamiltonian approach to analyze the problem. In addition to its simplicity this approach has the advantage that it also permits an elementary study of convergence of the equilibrium strategy if the planning horizon expands. Like in the theory on optimal control it turns out that under some conditions it can be shown that this strategy converges. One nice property of this converged solution is, as we will see, that it is rather easy to calculate and much easier

to implement than any finite planning horizon equilibrium solution. One would expect that this (converged) solution also solves the problem if the parties consider an infinite-planning horizon. Remarkably, however, the author was not able to trace a reference in literature dealing with this subject in a rigorous mathematical way. Particularly in the economic literature one either sticks to considering the limiting behaviour of the above mentioned finite-planning horizon solution, or imposes some additional constraints (e.g. the no-Ponzi game condition (see e.g. van Aarle et al. (1995))) on the solution of this problem. Therefore, we will consider this problem in somewhat more detail here. Two remarkable points we will see are that it may happen that, first, though the problem may have a unique equilibrium strategy for an arbitrary finite-planning horizon, there may exist more than one equilibrium solution for the infinite-planning horizon case and, second, the limit of this unique finite-planning horizon equilibrium solution may be not a solution for the infinite-planning horizon problem. On the other hand we will see that it can be easily verified whether the limiting solution of the finite-planning horizon problem solves also the infinite-planning horizon case.

The outline of the paper is as follows. In section two we start by stating the problem analysed in this paper and show how both a necessary and sufficient condition, in terms of a rank condition on a matrix, can be derived for the existence of a unique open-loop Nash equilibrium using the Hamiltonian approach. Moreover, we present the relationship that exists between solvability of a set of Riccati differential equations and solvability of the problem. Furthermore we give some simple sufficient conditions guaranteeing solvability of these Riccati differential equations. Before we present the convergence results of the finite-planning horizon equilibrium solution in section 4, we first consider the algebraic equations associated with the set of Riccati differential equations, and their solutions. In section 3 we show how all solutions of these equations can be determined from the eigenstructure of a certain matrix M , and that the eigenvalues of the associated closed-loop system, obtained by applying the limiting equilibrium strategy, are completely determined by the eigenvalues of this matrix M . A number of the results presented in sections 3 and 4 are also reported by Abou-Kandil et al. (1993). The conditions under which they derive the results are however not always completely specified and their proofs are of a more analytic nature. Therefore we choose to give here a selfcontained exposition including their results. The results on the infinite-planning horizon case are discussed in section 5. In particular we show that if the participating parties discount their future objectives, then the finite-planning horizon equilibrium solution converges to a limit which is generically the unique solution to the infinite-planning horizon case, if the discount factor is large enough. Finally, in section 6 we study the scalar case which is of particular interest for many economic applications. We show that in the scalar case, under a mild regularity condition, everything works out fine. This, in the sense that in this case the finite-planning equilibrium solution can be obtained by solving the set of Riccati differential equations and that the equilibrium solution converges to a stationary stabilizing feedback policy which also solves the infinite-planning horizon problem. The paper ends with some concluding remarks.

II. The finite-planning horizon case

In this paper we consider the problem where two parties (henceforth called players) try to minimize their individual quadratic performance criterion. Each player controls a different set of inputs to a single system, described by a differential equation of arbitrary order. As already mentioned in the introduction we assume that both players have to formulate their strategy already at the moment the system starts to evolve and this strategy can not be changed once the system runs. So, the players have to minimize their performance criterion based on the information that they only know the differential equation and its initial state. We are looking now for combinations of pairs of strategies of both players which are secure against any attempt by one player to unilaterally alter his strategy. That is, for those pairs of strategies which are such that if one player deviates from his strategy he will only lose. In the literature on dynamic games this problem is well-known as the open-loop Nash non-zero-sum linear quadratic differential game (see e.g. Starr and Ho (1969), Simaan and Cruz (1973), Başar and Olsder (1982) or Abou-Kandil and Bertrand (1986)). Formally the system we consider is as follows:

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \quad x(0) = x_0, \quad (1)$$

where x is the n -dimensional state of the system, u_i is an m -dimensional (control) vector player i can manipulate, x_0 is the initial state of the system, A , B_1 , and B_2 are constant matrices of appropriate dimensions, and \dot{x} denotes the time derivative of x .

The performance criterium player $i = 1, 2$ aims to minimize is:

$$J_1(u_1, u_2) := \frac{1}{2} x(t_f)^T K_{1f} x(t_f) + \frac{1}{2} \int_0^{t_f} \{x(t)^T Q_1 x(t) + u_1(t)^T R_{11} u_1(t) + u_2(t)^T R_{12} u_2(t)\} dt,$$

and

$$J_2(u_1, u_2) := \frac{1}{2} x(t_f)^T K_{2f} x(t_f) + \frac{1}{2} \int_0^{t_f} \{x(t)^T Q_2 x(t) + u_1(t)^T R_{21} u_1(t) + u_2(t)^T R_{22} u_2(t)\} dt,$$

in which all matrices are symmetric and, moreover, both Q_i and K_{if} are semi-positive definite and R_{ii} are positive definite.

In this section we consider in detail the existence of a unique open-loop Nash equilibrium of this differential game. Due to the stated assumptions both cost functionals $J_i, i = 1, 2$, are strictly convex functions of u_i for all admissible control functions $u_j, j \neq i$ and for all x_0 . This implies that the conditions following from the minimum principle are both necessary and sufficient (see e.g. Başar and Olsder (1982, section 6.5)).

Minimization of the Hamiltonian

$$H_i = (x^T Q_i x + u_1^T R_{i1} u_1 + u_2^T R_{i2} u_2) + \psi_i^T (Ax + B_1 u_1 + B_2 u_2), \quad i = 1, 2$$

with respect to u_i yields the optimality conditions (see e.g. Başar and Olsder (1982) or Abou-Kandil and Bertrand (1986)):

$$u_1^*(t) = -R_{11}^{-1} B_1^T \psi_1(t) \quad (2)$$

$$u_2^*(t) = -R_{22}^{-1} B_2^T \psi_2(t), \quad (3)$$

where the n -dimensional vectors $\psi_1(t)$ and $\psi_2(t)$ satisfy

$$\dot{\psi}_1(t) = -Q_1 x(t) - A^T \psi_1(t), \text{ with } \psi_1(t_f) = K_{1f} x(t_f)$$

$$\dot{\psi}_2(t) = -Q_2 x(t) - A^T \psi_2(t), \text{ with } \psi_2(t_f) = K_{2f} x(t_f)$$

and

$$\dot{x}(t) = Ax(t) - S_1 \psi_1(t) - S_2 \psi_2(t); \quad x(0) = x_0.$$

In other words, the problem has a unique open-loop Nash equilibrium if and only if the differential equation

$$\frac{d}{dt} \begin{pmatrix} x(t) \\ \psi_1(t) \\ \psi_2(t) \end{pmatrix} = - \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix} \begin{pmatrix} x(t) \\ \psi_1(t) \\ \psi_2(t) \end{pmatrix}$$

with boundary conditions $x(0) = x_0$, $\psi_1(t_f) - K_{1f} x(t_f) = 0$ and $\psi_2(t_f) - K_{2f} x(t_f) = 0$, has a unique solution. Denoting the state variable $(x^T(t) \ \psi_1^T(t) \ \psi_2^T(t))^T$ by $y(t)$, we can rewrite this two-point boundary value problem in the standard form

$$\dot{y}(t) = -My(t), \text{ with } Py(0) + Qy(t_f) = (x_0^T \ 0 \ 0)^T, \quad (4)$$

where $M = \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}$, $P = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 & 0 \\ -K_{1f} & I & 0 \\ -K_{2f} & 0 & I \end{pmatrix}$

From (4) we have immediately that problem (1) has a unique open-loop Nash equilibrium if and only if

$$(P + Qe^{-Mt_f})y(0) = (x_0^T \ 0 \ 0)^T,$$

or, equivalently,

$$(Pe^{Mt_f} + Q)e^{-Mt_f}y(0) = (x_0^T \ 0 \ 0)^T, \quad (5)$$

is uniquely solvable for every x_0 .

Using the following notation:

$$H(t_f) := W_{11}(t_f) + W_{12}(t_f)K_{1f} + W_{13}(t_f)K_{2f},$$

with $W(t_f) = (W_{ij}(t_f)) \{i, j = 1, 2, 3; \ W_{ij} \in R^{n \times n}\} := \exp(Mt_f)$, elementary matrix analysis then shows that

Theorem 1:

The two-player linear quadratic differential game (1) has a unique open-loop Nash equilibrium for every initial state if and only if matrix $H(t_f)$ is invertible. Moreover, the open-loop Nash equilibrium solution as well as the associated state trajectory can be calculated from the linear two-point boundary value problem (4). \square

Next, consider the following set of coupled asymmetric Riccati-type differential equations:

$$\dot{K}_1 = -A^T K_1 - K_1 A - Q_1 + K_1 S_1 K_1 + K_1 S_2 K_2; \quad K_1(t_f) = K_{1f} \quad (6)$$

$$\dot{K}_2 = -A^T K_2 - K_2 A - Q_2 + K_2 S_2 K_2 + K_2 S_1 K_1; \quad K_2(t_f) = K_{2f} \quad (7)$$

Let $K_i(t)$ satisfy this set of Riccati equations and assume that player i uses the strategy $u_i(t) = -R_{ii}^{-1} B_i^T K_i(t) x(t)$ to control system (1).

Now, define $\psi_i(t) := K_i(t)x(t)$. Then, obviously $\dot{\psi}_i(t) = \dot{K}_i(t)x(t) + K_i(t)\dot{x}(t)$.

Substitution of \dot{K}_i from (6,7) and \dot{x} from (1) yields

$$\dot{\psi}_i = (-A^T K_i - Q_i)x = -A^T \psi_i - Q_i x.$$

So, the two-point boundary value problem (4) has a solution. This proves the following claim:

Theorem 2:

The two-player linear quadratic differential game (1) has a unique open-loop Nash equilibrium for every initial state provided the set of Riccati equations (6,7) has a solution. Moreover, the equilibrium strategies are then given by:

$$u_1^*(t) = -R_{11}^{-1} B_1^T K_1(t) \Phi(t, 0) x_0 \quad (8)$$

$$u_2^*(t) = -R_{22}^{-1} B_2^T K_2(t) \Phi(t, 0) x_0 \quad (9)$$

Here $\Phi(t, 0)$ satisfies the transition equation

$$\dot{\Phi}(t, 0) = (A - S_1 K_1 - S_2 K_2) \Phi(t, 0); \quad \Phi(t, t) = I$$

and $S_i = B_i R_{ii}^{-1} B_i^T, i = 1, 2$. \square

The following example shows that there exist situations where the set of Riccati differential equations (6,7) does not have a solution, whereas there exists an open-loop Nash equilibrium for the game.

Example 3:

Let $A = \begin{pmatrix} -1 & 0 \\ 0 & -5/22 \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $Q_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $R_{11} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1}$, and $R_{22} = 1$.

Now, choose $t_f = 0.1$. Then, numerical calculation shows

$$H(0.1) = \begin{pmatrix} 1.1155 & 0.0051 \\ 0.0051 & 1.0230 \end{pmatrix} + \begin{pmatrix} 0.1007 & 0.1047 \\ 0.0964 & 0.2002 \end{pmatrix} K_{1f} + \\ \begin{pmatrix} 0.1005 & 0 \\ 0.0002 & 0 \end{pmatrix} K_{2f} =: V * (I \ K_{1f} \ K_{2f})^T.$$

Now, choose $K_{1f} = \begin{pmatrix} 1 & h_1 \\ h_1 & h_1^2 + 1 \end{pmatrix}$, where $h_1 = \frac{-V(1,2)-V(2,3)-V(2,5)*10}{V(2,4)}$, and $K_{2f} = \begin{pmatrix} 10 & h_2 \\ h_2 & \frac{h_2^2+1}{10} \end{pmatrix}$, where $h_2 = \frac{-V(2,2)-V(2,3)*K_{1f}(1,2)-V(2,4)*K_{1f}(2,2)}{V(2,5)}$. Then, clearly, both K_{1f} and K_{2f} are positive definite whereas the last row of $H(0.1)$ contains, by construction, only zeros, i.e. $H(0.1) = \begin{pmatrix} 2.1673 & -752.6945 \\ 0 & 0 \end{pmatrix}$ is not invertible.

So, according to theorem 1 the problem has not for every initial state a unique open-loop Nash equilibrium, and therefore (see theorem 2) the corresponding set of Riccati differential equations has no solution.

Next consider $H(0.11)$. Numerical calculation shows that with the system parameters as chosen above, $H(0.11)$ is invertible. So, according to theorem 1 again, the game does have an open-loop Nash equilibrium for $t_f = 0.11$. However, since the set of Riccati differential equations can be rewritten as one autonomous vector differential equation, whose solutions are known to be shift invariant, it is clear that the corresponding set of Riccati differential equations can not have a solution for $t_f = 0.11$, since it has no solution for $t_f = 0.1$. \square

Note that the above theorems are in fact local results. That is, they just make statements concerning existence of an equilibrium strategy for a fixed point in time. As we will see in a moment, the existence of an equilibrium strategy for every point in time during some fixed time interval $[0, t_f]$ is equivalent to the existence of a solution to the set of Riccati differential equations (6,7) on this interval.

One part of this conjecture is rather immediate. Assume that we know that the set of Riccati differential equations has a solution on $[0, t_f]$, then due to the time-invariance property of these differential equations also a solution exists to this set of equations for every point $t_1 \in [0, t_f]$. So, according theorem 2 there will exist an open-loop equilibrium strategy at every point in $[0, t_f]$.

On the other hand, if the open-loop problem has a solution it follows immediately from theorem 1 and (5) that

$$y_0 = e^{Mt_f} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} H^{-1}(t_f)x_0.$$

Since $y(t) = e^{-Mt}y_0$, it follows that the entries of $y(t)$ can be rewritten as

$$x(t) = (I \ 0 \ 0)e^{M(t_f-t)} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} H^{-1}(t_f)x_0 \quad (10)$$

$$\psi_1(t) = (0 \ I \ 0)e^{M(t_f-t)} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} H^{-1}(t_f)x_0 \quad (11)$$

$$\psi_2(t) = (0 \ 0 \ I)e^{M(t_f-t)} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} H^{-1}(t_f)x_0 \quad (12)$$

Using the previously introduced notation for $H(t)$, we see that (10) can be rewritten as $x(t) = H(t_f - t)H^{-1}(t_f)x_0$.

Since by assumption (see theorem 1) the matrix $H(t_f - t)$ is invertible it follows that $H^{-1}(t_f)x_0 = H^{-1}(t_f - t)x(t)$.

Substitution of this expression into the equations for $\psi_i, i = 1, 2$, in (11,12) yields:

$$\psi_1(t) = G_1(t_f - t)H^{-1}(t_f - t)x(t) \text{ and} \quad (13)$$

$$\psi_2(t) = G_2(t_f - t)H^{-1}(t_f - t)x(t) \quad (14)$$

for some continuously differentiable matrix functions $G_i, i = 1, 2$. and $H^{-1}(\cdot)$. Now, denote $G_i(t_f - t)H^{-1}(t_f - t)$ by $K_i(t), i = 1, 2$. Then, from (13), (14) it follows that $\dot{\psi}_i = \dot{K}_i x + K_i \dot{x}, i = 1, 2$.

From (2,3) we have that $\psi_1(t)$ and $\psi_2(t)$ satisfy

$$\dot{\psi}_1(t) = -Q_1 x(t) - A^T \psi_1(t), \text{ with } \psi_1(t_f) = K_{1f} x(t_f),$$

$$\dot{\psi}_2(t) = -Q_2 x(t) - A^T \psi_2(t), \text{ with } \psi_2(t_f) = K_{2f} x(t_f)$$

and

$$\dot{x}(t) = Ax(t) - S_1 \psi_1(t) - S_2 \psi_2(t); x(0) = x_0.$$

Substitution of $\dot{\psi}_i$ and $\psi_i, i = 1, 2$ into these formulas yields

$$(\dot{K}_1 + A^T K_1 + K_1 A + Q_1 - K_1 S_1 K_1 - K_1 S_2 K_2)e^{Mt}x_0 = 0 \text{ with } (K_1(t_f) - K_{1f})e^{Mt_f}x_0 = 0, \text{ and}$$

$$(\dot{K}_2 + A^T K_2 + K_2 A + Q_2 - K_2 S_2 K_2 - K_2 S_1 K_1)e^{Mt}x_0 = 0 \text{ with } (K_2(t_f) - K_{2f})e^{Mt_f}x_0 = 0,$$

for arbitrarily chosen x_0 .

From this it follows that $K_i(t), i = 1, 2$ satisfy the set of Riccati differential equations (6,7). This proves the following result:

Theorem 4:

The following statements are equivalent:

- 1) For all $t \in [0, t_1]$ there exists a unique open-loop Nash equilibrium for the two-player linear quadratic differential game (1).
- 2) $H(t)$ is invertible for all $t \in [0, t_1]$.
- 3) The set of Riccati differential equations (6,7) has a solution on $[0, t_1]$. \square

The above theorem shows that for both computational purposes and for a better theoretical understanding of the open-loop problem it would be nice to have a global existence result for the set of Riccati differential equations (6,7). Up to now this is, however, an unsolved problem. Sufficient conditions reported in literature on this subject (like the assumption that $Q_2 = \alpha Q_1$, see e.g. Abou-Kandil et al. (1986)) usually satisfy either one of the following cases (see Feucht (1994)):

Proposition 5:

The set of Riccati differential equations (6,7) has a solution on $[0, t_f]$ if either one of the next conditions is satisfied:

- 1) Either, (1) there exist matrices S and $C_i, i = 1, 2$ such that: i) $S_i = SC_i, i = 1, 2$, ii) $C_i A^T = A^T C_i, i = 1, 2$, iii) $S + S^T \geq 0$, iv) $C_1 Q_1 + C_2 Q_2 + (C_1 Q_1 + C_2 Q_2)^T \geq 0$, and v) $C_1 K_{1f} + C_2 K_{2f} + (C_1 K_{1f} + C_2 K_{2f})^T \geq 0$.
or, "dually", (2) there exist matrices Q and $D_i, i = 1, 2$ such that: i) $Q_i = D_i Q, i = 1, 2$, ii) $D_i A^T = A^T D_i, i = 1, 2$, iii) $Q + Q^T \geq 0$, iv) $S_1 D_1 + S_2 D_2 + (S_1 D_1 + S_2 D_2)^T \geq 0$, and v) $K_{1f} D_1 + K_{2f} D_2 + (K_{1f} D_1 + K_{2f} D_2)^T \geq 0$.
- 2) The model parameters satisfy the conditions: i) $S_i A^T = A S_i, i = 1, 2$, ii) $K_{1f} S_1 + S_1 K_{1f} + K_{2f} S_2 + S_2 K_{2f} - (A + A^T) \geq 0$, and iii) $A^2 + A^{T^2} + Q_1 S_1 + Q_2 S_2 + S_1 Q_1 + S_2 Q_2 \geq 0$. \square

If condition 1 of the proposition is satisfied and the matrix C_1 is invertible, matrix

M is similar to $\tilde{M} = \begin{pmatrix} -A & S & 0 \\ C_1 Q_1 + C_2 Q_2 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}$, (that is: $\tilde{M} = S M S^{-1}$, with

$S = \begin{pmatrix} I & 0 & 0 \\ 0 & C_1 & C_2 \\ 0 & 0 & I \end{pmatrix}$). This property can be helpful in calculating e.g. the eigenstructure

of matrix M . If e.g. matrix S is symmetric, $C_1 Q_1 + C_2 Q_2$ is both symmetric and invertible and M is dichotomically separable (see section 4) then this property implies that the finite game always has a solution which converges to a solution of the infinite-planning horizon problem, if the planning horizon expands to infinity.

III. The solutions for the algebraic Riccati equation

In this section we consider the set of solutions satisfying the set of so-called algebraic Riccati equations corresponding with (6,7)

$$\left. \begin{aligned} 0 &= -A^T K_1 - K_1 A - Q_1 + K_1 S_1 K_1 + K_1 S_2 K_2; \\ 0 &= -A^T K_2 - K_2 A - Q_2 + K_2 S_2 K_2 + K_2 S_1 K_1; \end{aligned} \right\} \quad (ARE)$$

MacFarlane (1963) and Potter (1966) independently discovered that there exists a relationship between the stabilizing solution of the algebraic Riccati equation and the eigenvectors of a related Hamiltonian matrix in linear quadratic regulator problems. We will follow their approach here and formulate similar results for our problem (1). In fact Abou-Kandil et al. (1993) already pointed out the existence of a similar relationship. One of their results is that if the planning horizon t_f in (1) tends to infinity, under some technical conditions on the matrix M , the solution of the above mentioned set of Riccati differential equations converges to a solution of the set of (ARE) which can be calculated from the eigenspaces of matrix M .

In this section we elaborate on the relationship between solutions of (ARE) and matrix M in detail. We present both necessary and sufficient conditions in terms of the matrix M under which (ARE) has (a) real solution(s). In particular we will see that all solutions of (ARE) can be calculated from the invariant subspaces of M and that the eigenvalues of the associated closed-loop system, obtained by applying the control $u_i^*(t) = -R_{ii}^{-1} B_i^T K_i(t) x(t)$, are completely determined by the eigenvalues of matrix M . As a corollary from these results we obtain both necessary and sufficient conditions for the existence of a stabilizing control of this type, a result which will be used in the next section.

In our analysis the set of all M -invariant subspaces plays a crucial role. Therefore we introduce a separate notation for this set:

$$\mathcal{M}^{inv} := \{ \mathcal{T} \mid M\mathcal{T} \subset \mathcal{T} \}.$$

It is well-known (see e.g. Lancaster and Tismenetsky (1985)) that this set contains only a finite number of (distinct) elements if and only if all eigenvalues of M have a geometric multiplicity one.

The set of possible solutions for the algebraic Riccati equation can, as will be shown in the next theorem, directly be calculated from the following collection of M -invariant subspaces:

$$\mathcal{K}^{pos} := \left\{ \mathcal{K} \in \mathcal{M}^{inv} \mid \mathcal{K} \oplus \text{Im} \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} = R^{3n} \right\}.$$

Note that elements in the set \mathcal{K}^{pos} can be calculated using the set of matrices

$$K^{pos} := \left\{ K \in R^{3n \times n} \mid \text{Im} K \oplus \text{Im} \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} = R^{3n} \right\}.$$

The exact result on how all solutions of (ARE) can be calculated is given in the next theorem. Its proof can be found in the appendix.

Theorem 6:

(ARE) has a real solution (K_1, K_2) if and only if $K_1 = YX^{-1}$ and $K_2 = ZX^{-1}$ for some

$$\mathcal{K} =: \text{Im} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in \mathcal{K}^{pos}.$$

Moreover, if the control functions $u_i^*(t) = -R_{ii}^{-1}B_i^TK_i\Phi(t)x_0$ are used to control the system (1), the spectrum of the closed-loop matrix $A - S_1K_1 - S_2K_2$ coincides with $\sigma(-M|_{\mathcal{K}})$. \square

From the above theorem a number of interesting properties concerning the solvability of (ARE) follow. First of all we observe that every element of \mathcal{K}^{pos} defines exactly one solution of (ARE). Furthermore, this set contains only a finite number of elements if and only if the geometric multiplicities of all eigenvalues of M is one. So, in that case we immediately conclude that (ARE) will have at most a finite number of solutions and that (ARE) will have no real solution if and only if \mathcal{K}^{pos} is empty.

Another conclusion which immediately follows from the above theorem is that

Corollary 7:

(ARE) will have a set of solutions (K_1, K_2) stabilizing the closed-loop system matrix $A - S_1K_1 - S_2K_2$ if and only if there exists an M invariant subspace \mathcal{K} in \mathcal{K}^{pos} such that $\text{Re } \lambda > 0$ for all $\lambda \in \sigma(M|_{\mathcal{K}})$. \square

To illustrate some of the above mentioned properties, reconsider example 3.

Example 3 (continued):

It can be shown analytically that both $\frac{-5}{22}$ and $\frac{-1}{2}$ are eigenvalues of M with multiplicity 2 and 1, respectively. Numerical calculations show that the other eigenvalues of M are $-1.8810, -0.1883$ and 1.7966 . Rearranging the eigenvalues as $\{\frac{-5}{22}, -1.8810, -0.1883, \frac{-1}{2}, 1.7966\}$ we have the following corresponding eigenspaces:

$$\begin{aligned} \mathcal{T}_1 &= \text{Span}\{T_{11} \ T_{12}\} \text{ where } T_{11} = (0 \ 0 \ 0 \ 0 \ 0 \ 1)^T, \text{ and} \\ &\quad T_{12} = (-0.2024 \ 0.6012 \ -0.2620 \ -0.0057 \ 0.5161 \ 0)^T; \\ \mathcal{T}_2 &= \text{Span}\{T_2\} \text{ where } T_2 = (-0.3726 \ -0.2006 \ 0.4229 \ 0 \ 0.6505 \ 0.4679)^T; \\ \mathcal{T}_3 &= \text{Span}\{T_3\} \text{ where } T_3 = (0.0079 \ -0.0234 \ 0.0097 \ 0 \ -0.0191 \ -0.9995)^T; \\ \mathcal{T}_4 &= \text{Span}\{T_4\} \text{ where } T_4 = (0.0580 \ -0.1596 \ 0.1160 \ 0 \ -0.2031 \ 0.9573)^T; \end{aligned}$$

and

$$\mathcal{T}_5 = \text{Span}\{T_5\} \text{ where } T_5 = (-0.7274 \ -0.1657 \ -0.2601 \ 0 \ -0.3194 \ -0.5232)^T.$$

According to theorem 6, the corresponding set of algebraic Riccati equations has at most $\binom{4}{2} = 6$ real solutions. Furthermore, there is no solution which stabilizes the closed-loop system matrix.

As an example consider $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} := (\mathcal{T}_2 \ \mathcal{T}_3)$. This yields the solution

$$K_1 = YX^{-1} = \begin{pmatrix} 0.4229 & 0.0097 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -0.3726 & 0.0079 \\ -0.2006 & -0.0234 \end{pmatrix}^{-1} \text{ and}$$

$$K_2 = ZX^{-1} = \begin{pmatrix} 0.6505 & -0.0191 \\ 0.4679 & -0.9995 \end{pmatrix} \begin{pmatrix} -0.3726 & 0.0079 \\ -0.2006 & -0.0234 \end{pmatrix}^{-1}.$$

The eigenvalues of the closed-loop system (1) using the control $u_i(t) = -R_{ii}^{-1}B_i^TK_ix(t)$ are $\{1.8810, 0.1883\}$. It is easily verified that the rank of the first two rows of every other candidate solution is also two, so we conclude that (ARE) has six solutions, none of which is stabilizing. \square

IV. Convergence results

As argued in the introduction, it is interesting to see how the open-loop equilibrium solution changes when the planning horizon t_f tends to infinity. To study convergence properties for problem (1), it seems reasonable to require that problem (1) has a properly defined solution for every finite planning horizon. Therefore in this section we will make the following well-posedness assumption (see theorem 4)

$$H(t_f) \text{ is invertible for all } t_f < \infty. \quad (15)$$

Furthermore, we will see that general convergence results can only be derived if the eigenstructure of matrix M satisfies an additional property, which we define first.

Definition 8:

M is called dichotomically separable if there exist subspaces V_1 and V_2 such that $MV_i \subset V_i, i = 1, 2, V_1 \oplus V_2 = \mathbb{R}^{3n}$, where $\dim V_1 = n$, $\dim V_2 = 2n$, and moreover $\operatorname{Re} \lambda > \operatorname{Re} \mu$ for all $\lambda \in \sigma(M|V_1), \mu \in \sigma(M|V_2)$. \square

From theorem 4 we know that (15) implies that to study the convergence of the open-loop Nash equilibrium solution we can restrict ourselves to the study of the set of Riccati differential equations (4-5) at time 0. We will denote the corresponding solutions of (4-5) by $K_i(0, t_f)$, respectively. So the question is under which conditions the solutions of this set of equations will converge if t_f increases. Note that $K_i(0, t_f)$ can be viewed as the solution $k(t)$ of an autonomous vector differential equation $\dot{k} = f(k)$, with $k(0) = k_0$ for some fixed k_0 , and where f is a smooth function. Elementary analysis shows then

that $K_i(0, t_f)$ converges to a limit \bar{k} only if this limit \bar{k} satisfies $f(\bar{k}) = 0$. Therefore, we immediately deduce from theorem 6 the following necessary condition for convergence.

Lemma 9:

$K_i(0, t_f)$ can only converge to a limit $\bar{K}_i(0)$ if the set \mathcal{K}^{pos} is nonempty. \square

Note that dichotomic separability of M implies that \mathcal{K}^{pos} is nonempty. On the other hand it is not difficult to construct an example where \mathcal{K}^{pos} is nonempty, whereas M is not dichotomically separable.

In the appendix we give an elementary proof of the following result (see also Abou-Kandil et al (1993, section 4))

Theorem 10:

Assume that the well-posedness assumption (15) holds.

Then, if M is dichotomically separable and $\text{Span} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \oplus V_2 = \mathbb{R}^{3n}$,

$$K_1(0, t_f) \rightarrow Y_0 X_0^{-1}, \text{ and } K_2(0, t_f) \rightarrow Z_0 X_0^{-1},$$

where X_0, Y_0, Z_0 are defined by (using the notation of definition 8) $V_1 =: \text{Span}(X_0^T \ Y_0^T \ Z_0^T)^T$.

Combination of the results from theorem 10 and corollary 7 yields then

Corollary 11:

If the planning horizon t_f in the differential game (1) tends to infinity, the unique open-loop Nash equilibrium solution converges to a stationary feedback strategy $u_i^*(t) = -R_{ii}^{-1} B_i^T K_i x(t)$, $i = 1, 2$, which stabilizes the associated closed-loop system, if the following conditions are satisfied:

1. all conditions mentioned in theorem 10
2. $\text{Re } \lambda > 0, \forall \lambda \in \sigma(M|V_1)$.

Moreover, these constant feedback matrices can be calculated from the eigenspaces of matrix M (see theorem 10). \square

V. The infinite-planning horizon case

In this section we assume that the performance criterium player $i = 1, 2$ likes to minimize, subject to the system dynamics (1), is:

$$\lim_{t_f \rightarrow \infty} J_i(u_1, u_2).$$

The information structure both players have at the beginning of the game is similar to the finite-planning horizon case. One additional assumption we make is that matrix Q_i is positive definite (instead of semi-positive definite) w.r.t. the controllable subspace $\langle A, B_i \rangle, i = 1, 2$. This has the immediate implication that, for well-posedness sake of the performance criteria, the equilibrium strategies we are looking for must be such that once they are applied, the state of the system converges to zero. In the appendix we prove the following theorem:

Theorem 12:

Let K_1 and K_2 be solutions of the algebraic Riccati equations (ARE) satisfying the additional constraint that the eigenvalues of $A - S_1 K_1 - S_2 K_2$ are all situated in the left half complex plane. Then, the strategy

$$u_i(t) = -R_i^{-1} B_i^T K_i x(t), i = 1, 2$$

is an open-loop Nash equilibrium strategy.

Moreover, the costs obtained by using this strategy for the players are

$$\int_0^\infty \{ (e^{(A-S_1 K_1 - S_2 K_2)t} x_0)^T (Q_i + K_i^T S_i K_i) e^{(A-S_1 K_1 - S_2 K_2)t} x_0 dt, i = 1, 2.$$

□

The next example shows that in general the infinite-planning horizon problem may have more than one open-loop Nash equilibrium, and that it is possible that the finite-planning horizon has no solution whereas a solution to the infinite-planning horizon exists.

Example 13:

Reconsider example 3 with matrix A replaced by $A = \begin{pmatrix} 1 & 0 \\ 0 & 5/22 \end{pmatrix}$.

Then, the eigenvalues of matrix M are $\{\frac{5}{22}, 1.8810, 0.1883, \frac{1}{2}, -1.7966\}$. Numerical calculation of the corresponding eigenspaces shows that the algebraic Riccati equations (ARE) have 3 stabilizing solutions. So, according the previous theorem, the infinite-planning horizon game has at least 3 open-loop Nash equilibria.

On the other hand it can be shown, by constructing final cost matrices K_{1f} and K_{2f} using the same procedure as in example 3, that matrix $H(t)$ at e.g. $t = 0.1$ is not always invertible. In other words, though the infinite planning horizon problem has solutions, a solution to the corresponding finite planning horizon problem may fail to exist. □

On the other hand one might hope that, if the finite planning horizon always has a unique solution, then this solution always converges to a solution of the infinite planning horizon case. That this conjecture is false is illustrated by the next example:

Example 14:

Reconsider example 3 (continued) with the following notation: $T := (T_{11} \ T_{12} \ T_2 \ T_3 \ T_4 \ T_5)$.

Using this notation we have that $e^{Mt} = T e^{Jt} T^{-1}$, where

$$J = \begin{pmatrix} \frac{-5}{22} & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{-5}{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.8810 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1883 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.7966 \end{pmatrix}. \text{ With } K_{1f} := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } K_{2f} := \begin{pmatrix} -\frac{T^{-1}(3,1)}{T^{-1}(3,5)} & -\frac{T^{-1}(3,2)}{T^{-1}(3,5)} \\ -\frac{T^{-1}(3,2)}{T^{-1}(3,5)} & 1 \end{pmatrix},$$

we have that both K_{1f} and K_{2f} are semi-positive definite. Numerical calculation shows that with these choices for the final cost, the determinant of $H(t)$ equals approximately $-0.02907 * e^{-0.6883t} + 2.419 * e^{1.6083t} - 1.390 * e^{1.2966t}$. From this we see that $H(t)$ is invertible for every positive t . So, the finite planning horizon problem has a unique equilibrium for every t_f . On the other hand, it is clear from theorem 10 that the equilibrium solution converges to the solution that can be calculated from $(X_0^T \ Y_0^T \ Z_0^T)^T := (T_3 \ T_5)$ (that is the eigenspaces corresponding with the two largest eigenvalues of M , -0.1883 and 1.7966). So, the converged equilibrium solution is not a "stabilizing solution". In other words the equilibrium solution does not converge to an equilibrium solution of the infinite game. \square

Next, we consider a class of systems for which we are able to derive both necessary and sufficient existence conditions. Following the variational (Hilbert space) approach Friedman took in (1971) (see also Lukes and Russell (1971) and Eisele (1982)) we prove in the appendix the following result:

Theorem 15:

Assume that matrix A is stable.

Then, \bar{u}_1, \bar{u}_2 is an open-loop Nash equilibrium solution for the infinite-planning horizon problem if and only if the following differential equation has a solution $v(t) =: (v_1^T(t) \ v_2^T(t) \ v_3^T(t))^T$, satisfying the property that $v_1(t), \bar{u}_1(t) := -R_{11}^{-1} B_1^T v_2(t)$ and $\bar{u}_2(t) := -R_{22}^{-1} B_2^T v_3(t)$ converge to zero if t tends to infinity:

$$\dot{v}(t) = -Mv(t), \tag{16}$$

with $v_1(0) = x_0$. \square

From this theorem we see in particular that existence of a Nash equilibrium may depend on the initial state of the system. That is, it may happen that for some parameter choices of the system matrices there are initial states of the system for which a Nash equilibrium exists, whereas for other initial states it may fail to exist.

We conclude this section by considering the case that both players discount their future

welfare loss. That is, the performance criterium player $i = 1, 2$ likes to minimize is:

$$\lim_{t_f \rightarrow \infty} J_i(u_1, u_2) := \frac{1}{2} \int_0^{t_f} e^{-rt} \{x(t)^T Q_i x(t) + u_i(t)^T R_{ii} u_i(t)\} dt,$$

where $r \geq 0$ is the discount factor.

Note from the previous analysis that dropping the cross term and the end-point cost in the cost-functional J_i has no influence on the obtained results. So the above formulation is as general as the one we considered before.

Now, introduce $\tilde{x}(t) := e^{-\frac{1}{2}rt} x(t)$ and $\tilde{u}_i(t) := e^{-\frac{1}{2}rt} u_i(t)$. Then, straightforward analysis shows that the above minimization problem can be rewritten as:

$$\min_{\tilde{u}_i} \lim_{t_f \rightarrow \infty} \frac{1}{2} \int_0^{t_f} \{\tilde{x}(t)^T Q_i \tilde{x}(t) + \tilde{u}_i(t)^T R_{ii} \tilde{u}_i(t)\} dt, \quad (17)$$

subject to

$$\dot{\tilde{x}} = (A - \frac{1}{2}rI)\tilde{x} + B_1 \tilde{u}_1 + B_2 \tilde{u}_2, \quad \tilde{x}(0) = x_0. \quad (18)$$

Obviously, all previously obtained results apply here with matrix M replaced by $\tilde{M} := \begin{pmatrix} -(A - \frac{1}{2}rI) & S_1 & S_2 \\ Q_1 & (A^T - \frac{1}{2}rI) & 0 \\ Q_2 & 0 & (A^T - \frac{1}{2}rI) \end{pmatrix}$. Using e.g. Geršgorin's theorem (see e.g. Lancaster et al. (1985, section 10.6)) it is clear that if r is large enough, matrix \tilde{M} will have $2n$ stable eigenvalues and n unstable ones. So, from theorem 15 we have the following result:

Theorem 16:

If the discount factor r is chosen large enough (see above) in the discounted optimization problem, then this problem has generically a unique Nash equilibrium. \square

VI. The scalar case and an economic example

We start this section by showing that the invertibility condition mentioned in theorem 4 is always satisfied if the dimensions of both the state and the input variables in system (1) equal one. This implies that for this kind of systems the usually stated assertion that the open-loop Nash strategy is given by (8,9) is correct and, moreover, that the associated Riccati equations yield the appropriate solution. To prove this result we first calculate the exponential of matrix M . To stress the fact that we are dealing with the scalar case, we will put the system parameters in lower case, so e.g. a instead of A .

Lemma 17:

Consider matrix M in (4). The exponential of matrix M , e^{Ms} , is given by

$$V \begin{pmatrix} e^{-\mu s} & 0 & 0 \\ 0 & e^{as} & 0 \\ 0 & 0 & e^{\mu s} \end{pmatrix} V^{-1}, \quad (19)$$

where

$$V = \begin{pmatrix} a + \mu & 0 & a - \mu \\ -q_1 & -s_2 & -q_1 \\ -q_2 & s_1 & -q_2 \end{pmatrix}$$

and its inverse

$$V^{-1} = \frac{1}{\det V} \begin{pmatrix} (s_1 q_1 + s_2 q_2) & s_1(a - \mu) & s_2(a - \mu) \\ 0 & -2q_2\mu & 2q_1\mu \\ -(s_1 q_1 + s_2 q_2) & -s_1(a + \mu) & -s_2(a + \mu) \end{pmatrix},$$

with the determinant of V , $\det V = 2\mu(s_1 q_1 + s_2 q_2)$, and $\mu = \sqrt{a^2 + s_1 q_1 + s_2 q_2}$.

Proof:

Straightforward multiplication shows that we can factorize M as $M = V \text{diag}(a, \mu, -\mu) V^{-1}$. So (see e.g. Lancaster et al (1985, theorem 9.4.3)), the exponential of matrix M , e^{Ms} , is as stated above. \square

Next consider the matrix $H(s)$ from theorem 4 for an arbitrarily chosen $s \in [0, t_f]$.

Obviously, $H(s) = (1 \ 0 \ 0) e^{Ms} \begin{pmatrix} 1 \\ k_{1f} \\ k_{2f} \end{pmatrix}$. Using the expressions in lemma 17 for V and

V^{-1} we find

$$H(s) = \frac{1}{\det V} [(s_1 q_1 + s_2 q_2) \{(\mu - a)e^{\mu s} + (a + \mu)e^{-\mu s}\} + (\mu^2 - a^2)(e^{\mu s} - e^{-\mu s})(s_1 k_{1f} + s_2 k_{2f})].$$

Clearly, $H(s)$ is positive for every $s \geq 0$. This implies in particular that $H(s)$ differs from zero for every $s \in [0, t_f]$, whatever $t_f > 0$ is. So from theorem 4 we now immediately have the following conclusion.

Theorem 18:

Problem (1) has a unique open-loop Nash equilibrium solution:

$$u_1^*(t) = -\frac{1}{r_{11}} b_1 k_1(t) x(t)$$

$$u_2^*(t) = -\frac{1}{r_{22}} b_2 k_2(t) x(t)$$

where $k_1(t)$ and $k_2(t)$ are the solutions of the coupled asymmetric Riccati-type differential equations

$$\begin{aligned} \dot{k}_1 &= -ak_1 - k_1 a - q_1 + k_1^2 s_1 + k_1 s_2 k_2; \quad k_1(t_f) = k_{1f} \\ \dot{k}_2 &= -ak_2 - k_2 a - q_2 + k_2^2 s_2 + k_2 s_1 k_1; \quad k_2(t_f) = k_{2f}. \end{aligned}$$

Here $s_i = \frac{1}{r_{ii}}b_i^2, i = 1, 2$. □

The next theorem shows that in the scalar case the equilibrium solution always converges.

Theorem 19:

Assume that $s_1q_1 + s_2q_2 > 0$.

Then, the open-loop Nash equilibrium solution from theorem 18 converges to the (stationary feedback) strategies:

$$u_1^*(t) = -\frac{1}{r_{11}}b_1k_1x(t)$$

$$u_2^*(t) = -\frac{1}{r_{22}}b_2k_2x(t)$$

where $k_1 = \frac{(a+\mu)q_1}{s_1q_1+s_2q_2}$ and $k_2 = \frac{(a+\mu)q_2}{s_1q_1+s_2q_2}$.

Moreover, these strategies are a solution to the infinite planning horizon open-loop problem.

Proof:

Since $s_1q_1 + s_2q_2 > 0$, it is clear from (19) that M is dichotomically separable. Furthermore we showed above that the well-posedness assumption is always satisfied in the scalar case. Note that $\mu > 0$, so according to corollary 11 the open-loop Nash strategies converge to a stationary feedback strategy whenever $k_{if}, i = 1, 2$, are such that $s_1q_1 + s_2q_2 + s_1(a - \mu)k_{1f} + s_2(a - \mu)k_{2f} \neq 0$.

Now consider the case that $s_1q_1 + s_2q_2 + s_1(a - \mu)k_{1f} + s_2(a - \mu)k_{2f} = 0$. To study this case, reconsider (23) and (24) for $t_f \rightarrow \infty$. Elementary spelling out of these formulas, using (19), shows that also in this case both $k_1(0, t_f)$ and $k_2(0, t_f)$ converge to the limits as advertised above. Using the results of theorem 12 this concludes the proof. □

We conclude this section by illustrating the computational advantages of our approach in an economic example. The example is taken from van Aarle et al. (1995). In this paper they analyze a differential game on government debt stabilization. They assume that government debt accumulation (\dot{d}) is the sum of interest payments on government debt ($rd(t)$) and primary fiscal deficits ($f(t)$) minus the seignorage (or the issue of base money) ($m(t)$):

$$\dot{d}(t) = rd(t) + f(t) - m(t), d(t_0) = d_0. \quad (20)$$

Here $d(t), f(t)$ and $m(t)$ are expressed as fractions of GDP and r represents the rate of interest on outstanding government debt minus the growth rate of output and is assumed to be exogenous. They assume that fiscal and monetary policies are controlled by different institutions, the fiscal authority and the monetary authority, respectively, which have different objectives. The objective of the fiscal authority is to minimize a

sum of time profiles of the primary fiscal deficit, base-money growth and government debt:

$$L^F(t_0) = \frac{1}{2} \int_{t_0}^{\infty} \{(f(t) - \bar{f})^2 + \eta(m(t) - \bar{m})^2 + \lambda(d(t) - \bar{d})^2\} e^{-\delta(t-t_0)} dt. \quad (21)$$

Whereas the monetary authorities set the growth of base money so as to minimize the loss function:

$$L^M(t_0) = \frac{1}{2} \int_{t_0}^{\infty} \{(m(t) - \bar{m})^2 + \kappa(d(t) - \bar{d})^2\} e^{-\delta(t-t_0)} dt. \quad (22)$$

Here $\frac{1}{\kappa}$ can be interpreted as a measure for the conservatism of the central bank w.r.t. the money growth. Furthermore all variables denoted with a bar are assumed to be fixed targets which are given a priori.

Introducing $x_1(t) := (d(t) - \bar{d})e^{-\frac{1}{2}\delta t}$, $x_2(t) := (r\bar{d} + \bar{f} - \bar{m})e^{-\frac{1}{2}\delta t}$, $u_1(t) := (f(t) - \bar{f})e^{-\frac{1}{2}\delta t}$ and $u_2(t) := (m(t) - \bar{m})e^{-\frac{1}{2}\delta t}$ the above game can be rewritten in our notation (1) with:

$$A = \begin{pmatrix} r - \frac{1}{2}\delta & 1 \\ 0 & -\frac{1}{2}\delta \end{pmatrix}, B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, B_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, Q_1 = \begin{pmatrix} \kappa & 0 \\ 0 & 0 \end{pmatrix}, Q_2 = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, R_{11} = 1 \text{ and } R_{22} = 1.$$

It is not difficult to see that the eigenvalues of M are: $\{-\frac{1}{2}\delta, -\frac{1}{2}\delta, \frac{1}{2}\delta, r - \frac{1}{2}\delta, l, -l\}$, where $l := \sqrt{\kappa + \lambda + (r - \frac{1}{2}\delta)^2}$. The corresponding eigenspaces are:

$$\begin{aligned} \mathcal{T}_1 &= \text{Span}\{T_1\} \text{ where } T_1 = (0 \ 0 \ 0 \ 0 \ 0 \ 1)^T, \\ \mathcal{T}_2 &= \text{Span}\{T_2\} \text{ where } T_2 = (0 \ 0 \ 0 \ 1 \ 0 \ 0)^T, \\ \mathcal{T}_3 &= \text{Span}\{T_3\} \text{ where } T_3 = (-(r - \delta)\delta \ \delta(\lambda + \kappa + r(r - \delta)) \ \kappa\delta \ \kappa \ \delta\lambda \ \lambda)^T, \\ \mathcal{T}_4 &= \text{Span}\{T_4\} \text{ where } T_4 = (0 \ 0 \ -r \ -1 \ r \ 1)^T, \\ \mathcal{T}_5 &= \text{Span}\{T_5\} \text{ where } T_5 = (\frac{1}{2}\delta - r + l \ 0 \ \kappa \ \frac{\kappa}{\frac{1}{2}\delta + l} \ \lambda \ \frac{\lambda}{\frac{1}{2}\delta + l})^T, \text{ and} \\ \mathcal{T}_6 &= \text{Span}\{T_6\} \text{ where } T_6 = ((\frac{1}{2}\delta - r - l)(\frac{1}{2}\delta - l) \ 0 \ \kappa(\frac{1}{2}\delta - l) \ \kappa \ \lambda(\frac{1}{2}\delta - l) \ \lambda)^T. \end{aligned}$$

So, the only stabilizing equilibrium strategy according theorem 12 is obtained by considering the eigenspaces corresponding to the eigenvalues $\frac{1}{2}\delta$ and l . This gives rise to $u_i = -B_i^T K_i x(t)$, with:

$$K_1 := \kappa \begin{pmatrix} \delta & 1 \\ 1 & \frac{1}{\frac{1}{2}\delta + l} \end{pmatrix} \begin{pmatrix} -(r - \delta)\delta & \frac{1}{2}\delta - r + l \\ \delta(\lambda + \kappa + r(r - \delta)) & 0 \end{pmatrix}^{-1} \text{ and } K_2 := \frac{\lambda}{\kappa} K_1.$$

In particular this implies that the equilibrium strategies satisfy the relationship $u_2(t) = -\frac{\lambda}{\kappa} u_1(t)$. Or, stated differently, $m(t) = \bar{m}(t) - \frac{\lambda}{\kappa}(f(t) - \bar{f}(t))$. Substitution of the equilibrium strategies into the system equation yields the closed-loop system

$$\dot{x}(t) = \begin{pmatrix} -l & p \\ 0 & -\frac{1}{2}\delta \end{pmatrix} x(t),$$

where $p = \frac{(\delta-r)(l-\frac{1}{2}\delta)}{\lambda+\kappa-r(\delta-r)}$. Note that we implicitly assumed here that $\lambda + \kappa + r(r - \delta)$ differs from zero; a technical assumption which is not crucial.

The advantages of this approach are clear. First, it gives more insight into the problem. That is, we obtain in an elementary way the optimal strategies and thus the closed-loop dynamics of the problem, which makes it e.g. additionally possible to state an exact condition (i.e. $\frac{1}{2}\delta - l < 0$) under which the analysis performed by van Aarle et al. holds and what happens if this condition is violated. Second, the approach can be straightforwardly generalized to multi-dimensional systems and is numerically easily implementable.

VII. Concluding remarks

In this paper we reconsidered the existence and asymptotic behaviour of open-loop Nash equilibrium solutions in the two-player linear quadratic game. Since the existing literature contains many inaccuracies w.r.t. this subject we gave an elementary selfcontained exposition. We analyzed the finite planning horizon problem starting from its basics: the Hamiltonian equations. We derived necessary and sufficient conditions for the existence of a unique open-loop Nash equilibrium solution in terms of a full rank condition on a modified fundamental matrix. We showed by means of an example that in general a solution to the set of associated differential Riccati equations may fail to exist whereas an open-loop Nash equilibrium solution exists. Furthermore, we showed that solvability of these Riccati equations is in fact related to existence of a Nash solution for every time during a fixed time interval. To study convergence of the open-loop equilibrium solution if the planning horizon is extended to infinity, we therefore studied the asymptotic behavior of the Riccati differential equations. To that end we first considered the existence of real solutions for the corresponding algebraic Riccati equations. We showed how every real solution to (ARE) can be calculated from the invariant subspaces of the

matrix $M = \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}$. Furthermore, we showed how the eigenvalues of the system, if the corresponding feedback control strategies are used in (1), can be obtained from the eigenvalues of this matrix.

In particular this approach makes it possible to conclude whether (ARE) has a real solution, and if so, how many solutions there are (there are always only a finite number of solutions if the geometric multiplicity of all eigenvalues of M is one) and which of them gives rise to control strategies that stabilize the closed-loop system. We noted that in general (ARE) will have more than one stabilizing solution.

The results on the existence of real solutions to (ARE) were used to show that if the dimension of the direct sum of the invariant subspaces corresponding with the n largest eigenvalues (counted again with algebraic multiplicities) equals n , then generically the solution to the Riccati differential equations converges to a solution which can be directly calculated from this direct sum. Moreover, if this solution stabilizes the closed-loop system it also solves the infinite-planning horizon problem.

The solution structure of the infinite-planning horizon problem turns out to be much more complicated than that of the finite-planning horizon case. Though, e.g., the finite planning-horizon case has a unique solution at every time it may both happen that the corresponding infinite game has no solution or more than one solution. In general we showed that the number of solutions depends on the initial state of the system. In case matrix A is stable we derived a both necessary and sufficient condition existence result. If a discounting factor is included in the performance function that is large enough, we reobtain uniqueness of the solution again (generically).

Since there are a number of applications which just involve scalar systems we concluded the paper by a detailed analysis of that case. We showed that for those systems, the unique open-loop Nash equilibrium solution can always be found by solving the associated set of Riccati differential equations, and that this solution converges to a stationary state feedback strategy, which stabilizes the associated closed-loop system if the planning horizon tends to infinity.

It will be clear that there are still a number of open problems in this area. In particular it remains a challenge for future research to get a better intuition and understanding why sometimes solutions to this open-loop problem exist whereas under some slight modifications they fail to exist. Finally we note that the obtained results can be straightforwardly generalized to the N player game.

An open problem remains to find general conditions on the system matrices which guarantee that the rank condition is satisfied.

Appendix

Proof of Theorem 6:

" \Rightarrow " Assume (K_1, K_2) solve (ARE). Then simple calculations show that

$$M \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} -A + S_1 K_1 + S_2 K_2 \\ Q_1 + A^T K_1 \\ Q_2 + A^T K_2 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} (-A + S_1 K_1 + S_2 K_2).$$

Now, introducing $X := I, Y := K_1$, and $Z := K_2$, we see that $M \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} J$,

for some matrix J and matrix X invertible, which completes this part of the proof.

" \Leftarrow " This part has been proved in a more general context by Meyer in (1976). However, since the last statement of the theorem can be immediately deduced from the following proof, we present this part of the proof here too.

Let $\mathcal{K} \in \mathcal{K}^{pos}$. Then there exist K_1 and K_2 such that $\mathcal{K} = Im \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix}$, and a matrix J

such that

$$M \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} J.$$

Spelling out the left hand side of this equation gives

$$\begin{pmatrix} -A + S_1 K_1 + S_2 K_2 \\ Q_1 + A^T K_1 \\ Q_2 + A^T K_2 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} J,$$

which immediately yields that $J = -A + S_1 K_1 + S_2 K_2$. Substitution of this equality into the right hand side of the equality shows then that $Q_1 + A^T K_1 = K_1(-A + S_1 K_1 + S_2 K_2)$ and $Q_2 + A^T K_2 = K_2(-A + S_1 K_1 + S_2 K_2)$, or stated differently, K_1, K_2 satisfy (ARE). This proves the second part of the theorem.

As already noted above, the last statement of the theorem concerning the spectrum of the matrix $-A + S_1 K_1 + S_2 K_2$ follows directly from the above arguments. If we

choose as a basis for \mathbb{R}^{3n} $\begin{pmatrix} I & 0 & 0 \\ K_1 & I & 0 \\ K_2 & 0 & I \end{pmatrix}$, matrix M has the block-triangular structure

$$\begin{pmatrix} -A + S_1 K_1 + S_2 K_2 & S_1 & S_2 \\ 0 & A^T - K_1 S_1 & -K_1 S_2 \\ 0 & -K_2 S_1 & A^T - K_2 S_2 \end{pmatrix}, \text{ which completes the proof. } \quad \square$$

Proof of Theorem 10

To study the convergence of $K_i(0, t_f)$ we reconsider (13) and (14). From these formulas we have that

$$K_1(0, t_f) = (0 \ I \ 0) e^{M t_f} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \left((I \ 0 \ 0) e^{M t_f} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \right)^{-1}, \text{ and} \quad (23)$$

$$K_2(0, t_f) = (0 \ 0 \ I) e^{M t_f} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \left((I \ 0 \ 0) e^{M t_f} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \right)^{-1}. \quad (24)$$

Now, choose $\begin{pmatrix} I & 0 & 0 \\ K_{1f} & I & 0 \\ K_{2f} & 0 & I \end{pmatrix}$ as a basis for \mathbb{R}^{3n} . Then, because

$$\text{Span} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \oplus V_2 = \mathbb{R}^{3n},$$

there exists an invertible matrix $V_{22} \in \mathbb{R}^{2n \times 2n}$ such that $V_2 = \text{Span} \begin{pmatrix} 0 \\ V_{22} \end{pmatrix}$.

Moreover, because M is dichotomically separable, there exist matrices J_1, J_2 such that

$$M = V \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} V^{-1},$$

where

$$V = \begin{pmatrix} X_0 & 0 \\ \begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix} & V_{22} \end{pmatrix},$$

and $\sigma(J_i) = \sigma(M|V_i)$, $i = 1, 2$.

Using this, we can rewrite $K_1(0, t_f)$ and $K_2(0, t_f)$ in (23,24) as $\tilde{G}_i(t_f)\tilde{H}^{-1}(t_f)$, $i = 1, 2$, where

$$\begin{aligned} \tilde{G}_1(t_f) &= (0 \ I \ 0)V e^{-\lambda_n t_f} \begin{pmatrix} e^{J_1 t_f} & 0 \\ 0 & e^{J_2 t_f} \end{pmatrix} V^{-1} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}, \\ \tilde{G}_2(t_f) &= (0 \ 0 \ I)V e^{-\lambda_n t_f} \begin{pmatrix} e^{J_1 t_f} & 0 \\ 0 & e^{J_2 t_f} \end{pmatrix} V^{-1} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}, \\ \tilde{H}(t_f) &= (I \ 0 \ 0)V e^{-\lambda_n t_f} \begin{pmatrix} e^{J_1 t_f} & 0 \\ 0 & e^{J_2 t_f} \end{pmatrix} V^{-1} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}. \end{aligned}$$

Here λ_n is the element of $\sigma(M|V_1)$ which has the smallest real part.

Next, consider $\tilde{G}_1(t_f) - Y_0 X_0^{-1} \tilde{H}(t_f)$.

Simple calculations show that this matrix can be rewritten as

$$e^{-\lambda_n t_f} (-Y_0 X_0^{-1} \ I \ 0)V \begin{pmatrix} e^{J_1 t_f} & 0 \\ 0 & e^{J_2 t_f} \end{pmatrix} V^{-1} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}, \quad (25)$$

Since $(-Y_0 X_0^{-1} \ I \ 0) \begin{pmatrix} X_0^T & Y_0^T & Z_0^T \end{pmatrix}^T = 0$, (25) equals

$$e^{-\lambda_n t_f} \begin{pmatrix} I & 0 \end{pmatrix} V_{22} e^{J_2 t_f} V_{22}^{-1} \begin{pmatrix} K_{1f} - Y_0 X_0^{-1} \\ K_{2f} - Z_0 X_0^{-1} \end{pmatrix}.$$

As $e^{-\lambda_n t_f} e^{J_2 t_f}$ converges to zero for $t_f \rightarrow \infty$, it is obvious now that $\tilde{G}_1(t_f) - Y_0 X_0^{-1} \tilde{H}(t_f)$ converges to zero for $t_f \rightarrow \infty$. Similarly it can be shown that also $\tilde{G}_2(t_f) - Z_0 X_0^{-1} \tilde{H}(t_f)$

converges to zero for $t_f \rightarrow \infty$. To conclude from this that $K_1(0, t_f) \rightarrow Y_0 X_0^{-1}$, and $K_2(0, t_f) \rightarrow Z_0 X_0^{-1}$, it suffices to show that $\tilde{H}^{-1}(t_f)$ remains bounded for $t_f \rightarrow \infty$. This follows, however, directly by spelling out $\tilde{H}(t_f)$ as

$$\tilde{H}(t_f) = e^{-\lambda_n t_f} X_0 e^{J_1 t_f} X_0^{-1}.$$

□

Proof of Theorem 12

Let K_1, K_2 be any pair of solutions satisfying the algebraic Riccati equations (ARE) and the additional constraint that the eigenvalues of $A - S_1 K_1 - S_2 K_2$ are all situated in the left half complex plane. We will show next that

$$\min_{u_1} \lim_{t_f \rightarrow \infty} J_1(u_1, u_2^*),$$

where $u_2^*(t) = -R_{22}^{-1} B_2^T K_2 e^{(A-S_1 K_1 - S_2 K_2)t}$, is obtained by choosing $u_1(t) = u_1^*(t) := -R_{11}^{-1} B_1^T K_1 e^{(A-S_1 K_1 - S_2 K_2)t}$. Since a similar reasoning shows that $\min_{u_2} \lim_{t_f \rightarrow \infty} J_2(u_1^*, u_2) \geq \lim_{t_f \rightarrow \infty} J_1(u_1^*, u_2^*)$, we have by definition that (u_1^*, u_2^*) is an open-loop Nash equilibrium.

To prove this claim, we first note that by substituting u_2^* into (1), we have that

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} [B_1 u_1(\tau) - S_2 K_2 e^{(A-S_1 K_1 - S_2 K_2)\tau} x_0] d\tau$$

Now, consider $\int_0^t e^{At} \frac{d}{d\tau} [e^{-A\tau} e^{(A-S_1 K_1 - S_2 K_2)\tau}] x_0 d\tau$. Evaluating this expression by on the one hand carrying out the differentiation w.r.t. τ and on the other hand calculating the integral yields the following equality:

$$\begin{aligned} \int_0^t e^{A(t-\tau)} (-A) e^{(A-S_1 K_1 - S_2 K_2)\tau} x_0 d\tau + \int_0^t e^{A(t-\tau)} (A - S_1 K_1 - S_2 K_2) e^{(A-S_1 K_1 - S_2 K_2)\tau} x_0 d\tau \\ = e^{(A-S_1 K_1 - S_2 K_2)t} x_0 - e^{At} x_0. \end{aligned}$$

Some elementary rewriting of this equality gives that:

$$\begin{aligned} e^{At} x_0 - \int_0^t e^{A(t-\tau)} S_2 K_2 e^{(A-S_1 K_1 - S_2 K_2)\tau} x_0 d\tau = \\ e^{(A-S_1 K_1 - S_2 K_2)t} x_0 + \int_0^t e^{A(t-\tau)} S_1 K_1 e^{(A-S_1 K_1 - S_2 K_2)\tau} x_0 d\tau. \end{aligned}$$

So, we can rewrite $x(t)$ as:

$$x(t) = e^{(A-S_1 K_1 - S_2 K_2)t} x_0 + \int_0^t e^{A(t-\tau)} B_1 \{ R_{11}^{-1} B_1^T K_1 e^{(A-S_1 K_1 - S_2 K_2)\tau} x_0 + u_1(\tau) \} d\tau$$

Therefore, using the notation $v(t) := u_1^*(t) + u_1(t)$,

$$\lim_{t_f \rightarrow \infty} J_1(u_1, u_2^*) = \int_0^\infty \{ (e^{(A-S_1K_1-S_2K_2)t} x_0 + \int_0^t e^{A(t-\tau)} B_1 v(\tau) d\tau)^T Q_1 (e^{(A-S_1K_1-S_2K_2)t} x_0 + \int_0^t e^{A(t-\tau)} B_1 v(\tau) d\tau) + u_1^T(t) R_{11} u_1(t) \} dt \quad (i)$$

Now, consider

$$s := \int_0^\infty \left(\int_0^t e^{A(t-\tau)} B_1 v(\tau) d\tau \right)^T Q_1 e^{(A-S_1K_1-S_2K_2)t} x_0 - v^T(t) B_1^T K_1 e^{(A-S_1K_1-S_2K_2)t} x_0 \right) dt.$$

Since, by assumption, K_1 and K_2 satisfy (ARE) we can rewrite Q_1 as:

$$Q_1 = -A^T K_1 - K_1 (A - S_1 K_1 - S_2 K_2)$$

Substitution into s yields:

$$\begin{aligned} s &= - \int_0^\infty \left(\int_0^t e^{A(t-\tau)} B_1 v(\tau) d\tau \right)^T (A^T K_1 + K_1 (A - S_1 K_1 - S_2 K_2)) e^{(A-S_1K_1-S_2K_2)t} x_0 \\ &\quad + v^T(t) B_1^T K_1 e^{(A-S_1K_1-S_2K_2)t} x_0 \right) dt \\ &= - \int_0^\infty \frac{d}{dt} \left(\int_0^t e^{A(t-\tau)} B_1 v(\tau) d\tau \right)^T K_1 e^{(A-S_1K_1-S_2K_2)t} x_0 dt \\ &= - \left(\lim_{t_f \rightarrow \infty} \int_0^{t_f} e^{A(t_f-\tau)} B_1 v(\tau) d\tau \right)^T K_1 e^{(A-S_1K_1-S_2K_2)t_f} x_0. \end{aligned}$$

Due to our assumption that all eigenvalues of $A - S_1 K_1 - S_2 K_2$ are stable and the fact that $J_1(u_1^*, u_2^*) < \infty$, which implies that we may assume without loss of generality (note that $Q_1 > 0 \forall x \in \langle A|B_1 \rangle$ in J_1) that $\lim_{t_f \rightarrow \infty} \int_0^{t_f} e^{A(t_f-\tau)} B_1 v(\tau) d\tau$ exists, we have from the last equation that $s = 0$. Rewriting the equation gives:

$$\int_0^\infty \left(\int_0^t e^{A(t-\tau)} B_1 v(\tau) d\tau \right)^T Q_1 e^{(A-S_1K_1-S_2K_2)t} x_0 dt = \int_0^\infty v^T(t) B_1^T K_1 e^{(A-S_1K_1-S_2K_2)t} x_0 dt.$$

The proof of the theorem is now completed by completion of squares. Substitution of the last expression into the formula (i) for J_1 shows that we can rewrite J_1 as:

$$\begin{aligned} &\int_0^\infty \{ (e^{(A-S_1K_1-S_2K_2)t} x_0)^T Q_1 e^{(A-S_1K_1-S_2K_2)t} x_0 + 2v^T(t) R_{11} u_1^*(t) + u_1^T(t) R_{11} u_1(t) + \\ &\quad \left(\int_0^t e^{A(t-\tau)} B_1 v(\tau) d\tau \right)^T Q_1 \int_0^t e^{A(t-\tau)} B_1 v(\tau) d\tau \} dt \\ &= \int_0^\infty \{ (e^{(A-S_1K_1-S_2K_2)t} x_0)^T Q_1 e^{(A-S_1K_1-S_2K_2)t} x_0 + (u_1(t) + u_1^*(t))^T R_{11} (u_1(t) + u_1^*(t)) + \\ &\quad (u_1^*(t))^T R_{11} u_1^*(t) + \left(\int_0^t e^{A(t-\tau)} B_1 (u_1^*(\tau) + u_1(\tau)) d\tau \right)^T Q_1 \int_0^t e^{A(t-\tau)} B_1 (u_1^*(\tau) + u_1(\tau)) d\tau \} dt. \end{aligned}$$

Using standard arguments it follows now immediately from this formula that $\lim_{t_f \rightarrow \infty} J_1(u_1, u_2^*)$ is minimal by choosing $u_1(t) = u_1^*(t)$. Moreover, its minimal value is $\int_0^\infty \{ (e^{(A-S_1 K_1 - S_2 K_2)t} x_0)^T (Q_1 + K_1^T S_1 K_1) e^{(A-S_1 K_1 - S_2 K_2)t} x_0 \} dt$. \square

Proof of Theorem 15:

Suppose that \bar{u}_1, \bar{u}_2 are a Nash solution. That is,

$$J_1(u_1, \bar{u}_2) \geq J_1(\bar{u}_1, \bar{u}_2) \text{ and } J_2(\bar{u}_1, u_2) \geq J_2(\bar{u}_1, \bar{u}_2). \quad (26)$$

Then, for any control function $w(t)$ for $u(t)$ and for any real number ϵ

$$J_1(\epsilon) := J_1(\bar{u}_1 + \epsilon w, \bar{u}_2) \geq J_1(\bar{u}_1, \bar{u}_2) \text{ and } J_2(\epsilon) := J_2(\bar{u}_1, \bar{u}_2 + \epsilon w) \geq J_2(\bar{u}_1, \bar{u}_2). \quad (27)$$

Let $\bar{x}(t)$ and $\bar{x}_\epsilon(t)$ be the solutions to 1 corresponding to the controls (\bar{u}_1, \bar{u}_2) and $(\bar{u}_1 + \epsilon w, \bar{u}_2)$, respectively. Then it is easily verified that

$$\bar{x}_\epsilon(t) = \bar{x}(t) + \epsilon \int_0^t e^{A(t-s)} B_1 w(s) ds. \quad (28)$$

So, under the assumption that the applied control stabilizes the system, $J_1(\epsilon)$ can be rewritten as:

$$\begin{aligned} & \frac{1}{2} \int_0^\infty \{ (\bar{x}(t) + \epsilon \int_0^t e^{A(t-s)} B_1 w(s) ds)^T Q_1 (\bar{x}(t) + \epsilon \int_0^t e^{A(t-s)} B_1 w(s) ds) + \\ & (u_1(t) + \epsilon w(t))^T R_{11} (u_1(t) + \epsilon w(t)) + u_2^T(t) R_{12} u_2(t) \} dt. \end{aligned}$$

From (27) we get

$$\left. \frac{dJ_i(\epsilon)}{d\epsilon} \right|_{\epsilon=0} = 0, i = 1, 2. \quad (29)$$

Evaluation of these expressions yields:

$$\int_0^\infty \{ (\int_0^t e^{A(t-s)} B_1 w(s) ds)^T Q_1 \bar{x}(t) + w^T(t) R_{11} \bar{u}_1(t) \} dt = 0.$$

Or, interchanging the order of integration in the first part:

$$\int_0^\infty \{ \int_s^\infty (e^{A(t-s)} B_1 w(s))^T Q_1 \bar{x}(t) dt \} ds + \int_0^\infty w^T(t) R_{11} \bar{u}_1(t) dt = 0.$$

Which can be restated as:

$$\int_0^\infty \{ w^T(t) B_1^T e^{-A^T t} \int_t^\infty e^{A^T s} Q_1 \bar{x}(s) ds \} dt + \int_0^\infty w^T(t) R_{11} \bar{u}_1(t) dt = 0.$$

Since $w(t)$ is arbitrary, it follows that

$$\bar{u}_1(t) = -R_{11}^{-1} B_1^T \int_t^\infty e^{A^T (s-t)} Q_1 \bar{x}(s) ds \quad (30)$$

Similarly, it can be shown that $\bar{u}_2(t)$ necessarily satisfies:

$$\bar{u}_2(t) = -R_{22}^{-1} B_2^T \int_t^\infty e^{A^T(s-t)} Q_2 \bar{x}(s) ds \quad (31)$$

Note that for any control $w(t)$ for u_i and for any real number ϵ ,

$$\frac{d^2 J_i(\epsilon)}{d\epsilon^2} > 0. \quad (32)$$

Thus, we have that $\bar{u}_1(t)$, $\bar{u}_2(t)$ constitutes an equilibrium strategy if and only if $\bar{x}(t)$, $\bar{u}_1(t)$ and $\bar{u}_2(t)$ satisfy (1), (30) and (31) and converge to zero if t goes to infinity.

Next, we show that this statement is equivalent to the advertised result in the theorem. To that end we introduce the vector $v := (v_1^T \ v_2^T \ v_3^T)^T$ as follows: $v_1(t) := \bar{x}(t)$, $v_2(t) := \int_t^\infty e^{A^T(s-t)} Q_1 \bar{x}(s) ds$ and $v_3(t) := \int_t^\infty e^{A^T(s-t)} Q_2 \bar{x}(s) ds$. Then $\bar{u}_1 = -R_{11}^{-1} B_1^T v_2(t)$ and $\bar{u}_2 = -R_{22}^{-1} B_2^T v_3(t)$. Moreover, it is easily verified by differentiation of $v_2(t)$ and $v_3(t)$, that $v(t)$ satisfies

$$\dot{v}(t) = - \begin{pmatrix} -A & B_1 R_{11}^{-1} B_1^T & B_2 R_{22}^{-1} B_2^T \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix} v(t), \text{ with } v_1(0) = x_0. \quad (33)$$

Furthermore, since A is stable and $\bar{x}(t) \rightarrow 0$ if $t \rightarrow \infty$, elementary analysis shows that $v(t) \rightarrow 0$ if $t \rightarrow \infty$, which concludes the "only if" part of the theorem.

On the other hand, if v satisfies (33) with $v(t) \rightarrow 0$ if $t \rightarrow \infty$, we introduce $\bar{x}(t) := v_1(t)$, $\bar{u}_1 := -R_{11}^{-1} B_1^T v_2(t)$ and $\bar{u}_2 = -R_{22}^{-1} B_2^T v_3(t)$. Since by assumption both $v_1(t)$ and $v_2(t)$ converge to zero if t goes to infinity and matrix A is stable, again elementary analysis shows that $v_2(t) := \int_t^\infty e^{A^T(s-t)} Q_1 \bar{x}(s) ds$. A similar reasoning shows that $v_3(t) := \int_t^\infty e^{A^T(s-t)} Q_2 \bar{x}(s) ds$. So, obviously, with these notation $\bar{x}(t)$, $\bar{u}_1(t)$ and $\bar{u}_2(t)$ satisfy (1), (30) and (31), and converge to zero if t goes to infinity, which concludes the proof. \square

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